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Degree of Approximation by Monotone Polynomials I.

G. G. LORENTZ[1]

Department of Mathematics, The University of Texas, Austin, Texas 78712

AND

K. L. ZELLER

Mathematisches Institut der Universität, 74 Tübingen, Germany

In late years, some attention has been attracted to the estimation of the degree of approximation of a continuous function by *monotone* polynomials, and more generally, by polynomials P_n of degree (at most) n, with a prescribed sign of the derivative $P_n^{(k)}(x)$ for some given k. See Shisha [3], Roulier [2]. Here we treat only the first case. (One can show uniqueness of the polynomial of best approximation in this case.) Let F be an increasing continuous function on $[-1, +1]$, let $E_n(F)$ be the degree of approximation of F by polynomials P_n of degree n. Let $E_n{}^*(F)$ denote the degree of approximation of F by *increasing* polynomials of degree n. We show that $E_n{}^*(F)$ satisfies an inequality of Jackson's type, even in its sharpened form (see Theorem 2). The proof will be conducted by means of trigonometric approximation. A continuous 2π-periodic function on $[-\pi, \pi]$ will be called *bell-shaped*, if it is even and if it decreases on $[0, \pi]$. By $\omega(f, h) = \omega(h)$ we denote the modulus of continuity of f, by $C_1, C_2, \ldots,$ absolute constants.

THEOREM 1. *There exists a constant C with the following property. For each bell-shaped function f, one can find a bell-shaped trigonometric polynomial T_n for which*

$$|f(x) - T_n(x)| \leqslant C\omega\left(f, \frac{1}{n}\right). \tag{1}$$

Proof. Let J_n be the Jackson integral of f,

$$J_n(x) = J_n(f, x) = \int_{-\pi}^{\pi} K_n(x - t) f(t)\, dt,$$

$$K_n(t) = \lambda_n^{-1} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^4,$$

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where λ_n is a normalizing constant. We put

$$g(t)=g(f,t)=\begin{cases} f\left(\frac{\pi k}{n}\right)=c_k, & \text{for } \frac{\pi k}{n}\leqslant t<\frac{\pi(k+1)}{n}, \quad k=0,1,\ldots,n, \\ g(-t), & \text{for } -\pi\leqslant t\leqslant 0. \end{cases} \tag{2}$$

Then g is even and $|f(t)-g(t)|\leqslant\omega(\pi/n)$. Let $T_n(t)=J_n(g,t)$. Then T_n is even, and because of the inequality

$$\begin{aligned}|f(t)-T_n(t)| &\leqslant |f(t)-J_n(f,t)|+|J_n(f,t)-J_n(g,t)| \\ &\leqslant C_1\,\omega\left(f,\frac{1}{n}\right)+\|f-g\|\leqslant C_2\,\omega\left(f,\frac{1}{n}\right),\end{aligned} \tag{3}$$

it is sufficient to show that $T_n(x)$ is decreasing on $(0,\pi)$.

Let a_k, $k=0,\ldots,n-1$, be defined by

$$c_k=a_k+\cdots+a_{n-1}, \qquad k=0,\ldots,n-1.$$

Then $a_k\geqslant 0$, and

$$T_n(x)=\sum_{k=0}^{n-1}a_k\int_{-\pi(k+1)/n}^{\pi(k+1)/n}K_n(x-t)\,dt=\sum_{k=1}^{n}a_{k-1}\int_{-\pi k/n}^{\pi k/n}K_n(x-t)\,dt.$$

Hence it is sufficient to show that the following functions are decreasing on $(0,\pi)$:

$$\phi_k(x)=\int_{-\pi k/n}^{\pi k/n}K_n(x-t)\,dt=\int_{x-(\pi k/n)}^{x+(\pi k/n)}K_n(t)\,dt.$$

But

$$\begin{aligned}\phi_k'(x) &= K_n\left(x+\frac{\pi k}{n}\right)-K_n\left(x-\frac{\pi k}{n}\right) \\ &= \lambda_n^{-1}\sin^4\frac{nx+\pi k}{2}\left\{\frac{1}{\sin^4\frac{1}{2}\left(x+\frac{\pi k}{n}\right)}-\frac{1}{\sin^4\frac{1}{2}\left(x-\frac{\pi k}{n}\right)}\right\}\leqslant 0.\end{aligned}$$

This follows from the following inequality:

$$\sin(\alpha+\beta)\geqslant|\sin(\alpha-\beta)| \qquad \text{if } 0\leqslant\alpha,\beta=\frac{\pi}{2}.$$

In fact, $\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\sin\beta\cos\alpha\geqslant 0$, and $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta\geqslant 0$.

THEOREM 2. *There is a constant C_0 with the following property. If F is an increasing function on* $[-1, +1]$, *then there exists a sequence of polynomials P_n, increasing on* $[-1, +1]$, *such that*

$$|F(x) - P_n(x)| \leqslant C_0\, \omega(F, \Delta_n(x)), \qquad \Delta_n(x) = \max\left(\frac{\sqrt{1-x^2}}{n}, \frac{1}{n^2}\right), \quad n = 1, 2, \ldots \tag{4}$$

Proof. The function $f(t) = F(\cos t)$ is bell-shaped. For this function, we prove similarly to (1) of Theorem 1,

$$|f(t) - T_n(t)| \leqslant C\omega(F, \delta_n), \qquad \delta_n(t) = \max\left(\frac{|\sin t|}{n}, \frac{1}{n^2}\right), \tag{5}$$

with some bell-shaped T_n. We have ([*1*], p. 68), if $J_n(t)$ is the Jackson polynomial of the function f,

$$|F(\cos t) - J_n(t)| \leqslant C_1\, \omega(F, \delta_n). \tag{6}$$

We put $g(t) = g(f, t)$; then, as before, $J_n(g, t)$ is bell-shaped, while

$$|f(t) - g(f, t)| \leqslant \omega(F, h), \qquad h = \max|\cos t_1 - \cos t|,$$

where (if, for example, $t > 0$), t_1 is given by $t_1 = \pi k/n$, $t_1 \leqslant t < t_1 + \pi/n$ for some k. Hence

$$h \leqslant 2 \sin\frac{\pi}{2n} \sin\frac{\pi k}{2n} \leqslant C\frac{1}{n} \sin t \leqslant C\delta_n(t).$$

From (3) and (6) we obtain

$$\begin{aligned} |F(\cos t) - T_n(t)| &\leqslant C_1\, \omega(F, \delta_n) + \|f - g\| \\ &\leqslant C_2\, \omega(F, \delta_n). \end{aligned}$$

This proves (5). We obtain (4) from this by means of the substitution $x = \cos t$.

It is not known whether there exists an absolute constant A for which

$$E_n^*(F) \leqslant AE_n(F)$$

for each increasing F on $[-1, +1]$. Here is a partial substitute:

THEOREM 3. *There is a constant A so that if F is an increasing function on* $[-1, +1]$, *and if $E_n(F) \leqslant \omega(1/n)$, where ω is some modulus of continuity, then*

$$E_n^*(F) \leqslant A\frac{1}{n}\sum_{k=1}^{n} \omega\left(\frac{1}{k}\right), \quad n = 1, 2, \ldots. \tag{7}$$

(For a characterization of a modulus of continuity see [*1*, p. 43].) Relation (7) follows by combining Theorem 2 with the known inequality ([*1*], p. 73)

$$\omega(F, h) \leqslant Mh \sum_{1 < k < h^{-1}} \omega\left(\frac{1}{k}\right),$$

valid for each function F that satisfies $|F(x) - P_n(x)| \leqslant \omega(\Delta_n(x))$.

References

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3. O. Shisha, Monotone approximation. *Pacific J. Math.* **15** (1965), 667–671.